13.4 Green's Theorem

13.4.1 Green's Theorem

Definition

A Cartesian region that is both horizontally and vertically simple (Type I and Type II) is called a **simple region**.

Definition

A simple closed curve has **positive orientation** if it is traversed counterclockwise exactly once. In this case, the enclosed region should always be to the *left* while traversing the path.

Green's Theorem

Let D be the Cartesian region bounded by a positively oriented, piecewise-smooth, simple closed curve C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\ dx + Q\ dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$$

Note: So we have a choice of which integral we'd like to evaluate! If you see it, choose which integral is more reasonable to evaluate.

Convention

When writing a line integral over the curve C, we may use $\oint_C$ rather than $\int_C$ in the case that C is a closed loop. Computationally, $\oint$ and $\int$ represent the exact same thing. Conceptually, $\oint$ is a symbol that further states that the curve being integrated over is closed (and thus encloses a region).

Example 1. Evaluate $\int_C x^4\ dx + xy\ dy$, where C is the triangular path traveling from the origin to $(1,0)$, then to $(0,1)$, then back to the origin.

Corollary

Let D be the Cartesian region bounded by a positively oriented, piecewise-smooth, simple closed curve C. Then

$$\text{Area}(D) = \oint_C x\ dy = -\oint_C y\ dx = \frac{1}{2}\oint_C x\ dy - y\ dx$$

Proof:

Example 2. Find the area enclosed by the ellipse $\frac{x^2}{16} + \frac{y^2}{49} = 1$.

13.4.2 Green's Theorem for Finite Union of Regions

Theorem

If $D = D_1 \cup D_2$, where D_1, D_2 are simple with respective boundaries C_1, C_2, then

$$\int_{C_1 \cup C_2} P\ dx + Q\ dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA$$

Note: This extension of Green's Theorem allows us to both use Green's Theorem on the union of disjoint simple regions and also to cut up simple regions into smaller, disjoint simple regions.

Example 3. Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region between the circles of radii 1 and 2 centered at the origin in the first two quadrants.

Big Idea – using line integrals and Green's Theorem to detect Conservative Vector Fields: Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a vector field on an open simply-connected region E, where P, Q have continuous first-order partial derivatives. Suppose further that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ on E. Now, if C is any simple closed path in E, and D is the region enclosed by C, then

$$\oint \mathbf{F} \cdot d\mathbf{r} = \oint P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D 0\,dA = 0$$

Now, a curve that is not simple crosses itself at least once, and we can break these up into several simple curves (not necessarily positively oriented). Since each of these line integrals of $\mathbf{F}$ is 0, the sum of all of them is 0, and $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C. Hence, $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path-independent in D. Therefore, $\mathbf{F}$ is a conservative vector field.

Green's Theorem (Simple Case):
Proof