MTH 253
Final Review Key

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1. Determine whether the sequence $\{\tan n\}$ converges or diverges.

Solution: Since tangent is periodic, it will continue to repeat itself forever. Thus, $\{\tan n\}$ does not converge. Therefore, $\{\tan n\}$ diverges as a sequence.

2. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+1)!}{2^n}$ converges or diverges. Justify your conclusion as specifically as possible.

Solution: The Divergence Test works, as $\lim_{n\to\infty} \frac{(-1)^{n+1}(2n+1)!}{2^n}$ does not exist.

The Ratio Test works, as

$$\begin{aligned}
\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| &= \lim_{n\to\infty} |\frac{(-1)^{(n+1)+1}(2(n+1)+1)!}{2^(n+1)} \cdot \frac{2^n}{(-1)^{n+1}(2n+1)!}| \\
&= \lim_{n\to\infty} |\frac{(2n+3)!2^n}{(2n+1)!2^{n+1}}| \\
&= \lim_{n\to\infty} |\frac{(2n+2)(2n+1)}{2}| \\
&= \infty
\end{aligned}$$

By the Ratio Test, since $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| \not\leq 1$ (more precisely, the limit DNE), the series diverges.

The Integral Test doesn't work since the associated function is not continuous (factorial is not continuous).

The Comparion Test may work, but we don't have a quick series to compare it against.

The Limit Comparison Test may work, but we don't have a quick series to compare it against.

The series is neither a geometric nor a p-series.

The Alternating Series Test is inconclusive since $\lim_{n\to\infty} \frac{(2n+1)!}{2^n} \neq 0$.

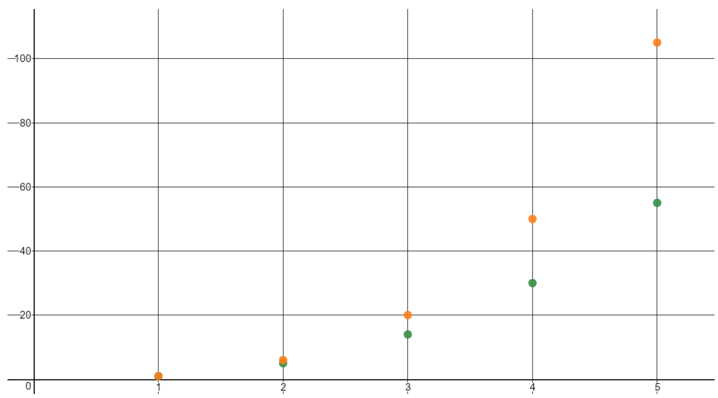
3. Write the first five terms of the sequence $\left\{\frac{n(n+1)(2n+1)}{6}\right\}$.

Solution: We substitute in $n = 1, 2, 3, 4, 5$ to obtain each term.

$$\begin{aligned}
a_1 &= \frac{(1)((1)+1)(2(1)+1)}{6} &&= 1 \\
a_2 &= \frac{(2)((2)+1)(2(2)+1)}{6} &&= 5 \\
a_3 &= \frac{(3)((3)+1)(2(3)+1)}{6} &&= 14 \\
a_4 &= \frac{(4)((4)+1)(2(4)+1)}{6} &&= 30 \\
a_5 &= \frac{(5)((5)+1)(2(5)+1)}{6} &&= 55
\end{aligned}$$

4. Graph the first five terms of the sequence $\left\{ \dfrac{n(n+1)(2n+1)}{6} \right\}$. Then graph the first five partial sums.

Solution: The first five terms of the sequence are graphed in green. The first five partial sums are graphed in orange.



5. Consider the sequence $\{a_n\} = \{128, -96, 72, -54, 40.5, \ldots\}$.
   (a) What kind of a sequence is this?
   (b) Write a formula for a_n, the nth term of the sequence.
   (c) Determine whether the sequence $\{a_n\}$ converges or diverges. If the sequence converges, what does it converge to?
   (d) Consider the series $\displaystyle\sum_{n=1}^{\infty} a_n$. Find the first three partial sums.
   (e) Find the exact value of the sum of the series $\displaystyle\sum_{n=1}^{\infty} a_n$.

Solution: Consider the ratio $\dfrac{a_{n+1}}{a_n}$ for $n = 1, 2, 3, 4, \ldots$. We get

$$\frac{-96}{128} = \frac{-3}{4} \quad \frac{72}{-96} = \frac{-3}{4} \quad \frac{-54}{72} = \frac{-3}{4} \quad \frac{40.5}{-54} = \frac{-3}{4}$$

We can therefore conclude that the sequence is **geometric**. This geometric series has **initial term** $a = 128$ and **common ratio** $r = \dfrac{-3}{4}$.

Geometric sequences have the form $a_n = ar^{n-1}$, so $a_n = 128\left(\frac{-3}{4}\right)^{n-1}$.

The sequence of terms converges to 0, since geometric sequences converge to 0 when $|r| < 1$ (a fancy way of saying $-1 < r < 1$) and diverge in all other cases.

The **geometric series** $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 128\left(\frac{-3}{4}\right)^{n-1}$ has partial sums found by adding up the first n terms. If we define s_n to be the nth partial sum, then

$$\begin{aligned} s_1 &= 128 \\ s_2 &= 128 + (-96) = 32 \\ s_3 &= 128 + (-96) + 72 = 104 \end{aligned}$$

A **geometric series** converges when $|r| < 1$. Since $|r| = |\frac{-3}{4}| < 1$, we can conclude that the series converges. Moreover, if a_n is geometric, then $\sum_{n=1}^{\infty} a_n = \frac{a}{1-r}$. In particular

$$\begin{aligned} \sum_{n=1}^{\infty} 128\left(\frac{-3}{4}\right)^{n-1} &= \frac{128}{1 - \frac{-3}{4}} \\ &= \frac{512}{7} \end{aligned}$$

6. Determine whether the series below converges or diverges. Justify your conclusion as specifically as possible.

$$1 - \frac{1}{2} + 1 - \frac{1}{4} + 1 - \frac{1}{8} + 1 - \frac{1}{16} + \cdots$$

Solution: We can represent this series by grouping pairs of terms together. That is

$$1 - \frac{1}{2} + 1 - \frac{1}{4} + 1 - \frac{1}{8} + 1 - \frac{1}{16} + \cdots = \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{8}\right) + \left(1 - \frac{1}{16}\right) + \cdots$$

By looking at this representation, we can write

$$\left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{8}\right) + \left(1 - \frac{1}{16}\right) + \cdots = \sum_{n=1}^{\infty}\left(1 - \frac{1}{2^n}\right)$$

Writing the series in this way is not necessary, but it does make analysis a bit easier.

The Divergence test works. Note $\lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 1 \neq 0$. Therefore, the series diverges by the Divergence Test.

The Ratio Test may work. Note

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2^n}}\right| \\ &= \frac{1-0}{1-0} \\ &= 1 \end{aligned}$$

It follows that the Ratio Test is inconclusive.

By the Integral Test, we have that $\displaystyle\sum_{n=1}^{\infty}\left(1-\frac{1}{2^n}\right)$ converges when $\displaystyle\int_1^{\infty}\left(1-\frac{1}{2^x}\right)\,dx$ converges, and diverges when they both diverge. Note

$$\begin{aligned}
\int_1^{\infty}\left(1-\frac{1}{2^x}\right)\,dx &= \lim_{t\to\infty}\int_1^{t}\left(1-2^{-x}\right)\,dx \\
&= \lim_{t\to\infty}\left[x+\frac{2^{-x}}{\ln 2}\right]_1^t \\
&= \lim_{t\to\infty}\left[t+\frac{2^{-t}}{\ln 2}-\left(1+\frac{2^{-1}}{\ln 2}\right)\right] \\
&= \lim_{t\to\infty}(t)+0-1+\frac{1}{2\ln 2} \\
&= \infty
\end{aligned}$$

It follows that since $\displaystyle\int_1^{\infty}\left(1-\frac{1}{2^x}\right)\,dx$ diverges, $\displaystyle\sum_{n=1}^{\infty}\left(1-\frac{1}{2^n}\right)$ diverges by the Integral Test.

We can compare this series to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2}$, since $1-\dfrac{1}{2^n}>\dfrac{1}{2}$ for all $n\geq 2$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2}$ diverges (Divergence Test), $\displaystyle\sum_{n=1}^{\infty}\left(1-\frac{1}{2^n}\right)$ diverges by the Comparison Test.

We can compare the series to $\displaystyle\sum_{n=1}^{\infty}1$ with the Limit Comparison Test. Note

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{1-\frac{1}{2^n}} &= \frac{1}{1-0} \\
&= 1
\end{aligned}$$

Since $1>0$, both series diverge by the Limit Comparison Test.

The series is neither a geometric nor a p-series.

The Alternating Series Test does not apply since non-alternating terms are not decreasing. That is, $1>\frac{1}{2}\not>1$.

7. Determine whether the series below converges or diverges. Justify your conclusion as specifically as possible.

$$\sum_{n=1}^{\infty}n^2e^{-n^3}$$

Solution: The Divergence Test is inconclusive, since $\displaystyle\lim_{n\to\infty}n^2e^{-n^3}=0$.

The Ratio Test works. Note

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)^2}{e^{(n+1)^3}}\cdot\frac{e^{n^3}}{n^2}\right| \\
&= \lim_{n\to\infty}\left|\frac{n^2+2n+1}{n^2}\cdot\frac{e^{n^3}}{e^{(n+1)^3}}\right| \\
&= \lim_{n\to\infty}\left|\frac{n^2+2n+1}{n^2}\cdot e^{n^3-(n+1)^3}\right| \\
&= \lim_{n\to\infty}\frac{n^2+2n+1}{n^2}\cdot\lim_{n\to\infty}e^{n^3-n^3-3n^2-3n-1} \\
&= (1)\cdot(0) \\
&= 0<1
\end{aligned}$$

Since the limit is less than 1, the series converges by the Ratio Test.

By the Integral Test, we have that $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges when $\int_1^{\infty} x^2 e^{-x^3}\, dx$ converges. Consider

$$\int_1^{\infty} x^2 e^{-x^3}\, dx = \lim_{t\to\infty} \int_1^{t} x^2 e^{-x^3}\, dx$$

Using the substitution $u = x^3$, we get $du = 3x^2\, dx$, and

$$\begin{aligned}\lim_{t\to\infty} \int_1^{t} x^2 e^{-x^3}\, dx &= \lim_{t\to\infty} \int_1^{t} \frac{1}{3} e^{-u}\, du \\ &= \lim_{t\to\infty} \frac{1}{3}\left[-e^{-u}\right]_1^t \\ &= \lim_{t\to\infty} \frac{1}{3}\left[-e^{-t} + e^{-1}\right] \\ &= \frac{1}{3}\left[0 + e^{-1}\right]\end{aligned}$$

Because our integral converges, the series $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges, as well by the Integral Test.

The Comparison Test may work, but an immediate comparison is not obvious. However, $n^2 e^{-n^3} < e^{-n}$ for $n > 1$, and $\sum_{n=1}^{\infty} e^{-n}$ is a geometric series with common ratio e^{-1}, and $|e^{-1}| < 1$, so it converges. By the Comparison Test, the series converges.

The Limit Comparison Test will work with the same series listed above.

The series is neither a p-series nor geometric.

The series is not an Alternating Series.

8. Consider the function $f(x) = \dfrac{x^3}{x^3+2}$.
   (a) Express $f(x)$ as a power series.
   (b) Determine the radius of convergence for the power series you found in (a).
   (c) Determine the interval of convergence for the power series you found in (a).
   (d) Find $f'(x)$ by differentiating the power series you found in (a) term-by-term.
   (e) Find an antiderivative for f by integrating the power series you found in (a) term-by-term.

Solution: We want to manipulate $\dfrac{x^3}{x^3+2}$ to have the form $\dfrac{\Box}{1-\Box}$, since

$$\frac{\Box}{1-\Box} = \sum_{n=0}^{\infty} \Box^n$$

Consider that

$$\begin{aligned}\frac{x^3}{x^3+2} &= x^3 \frac{1}{x^3+2} \\ &= x^3 \frac{1}{2-(-x^3)} \\ &= x^3 \frac{1}{2(1-\frac{-x^3}{2}} \\ &= \frac{x^3}{2} \frac{1}{1-\frac{-x^3}{2}}\end{aligned}$$

Since the last factor has the form $\frac{\square}{1-\square}$, it follows that

$$\frac{1}{1-\frac{-x^3}{2}} = \sum_{n=0}^{\infty} \left(\frac{-x^3}{2}\right)^n$$

Therefore,

$$\begin{aligned} f(x) &= \frac{x^3}{x^3+2} \\ &= \frac{x^3}{2} \frac{1}{1-\frac{-x^3}{2}} \\ &= \frac{x^3}{2} \sum_{n=0}^{\infty} \left(\frac{-x^3}{2}\right)^n \end{aligned}$$

Now, $\sum_{n=0}^{\infty} \square^n$ converges when $|\square| < 1$. Since $f(x) = \frac{x^3}{2} \sum_{n=0}^{\infty} \left(\frac{-x^3}{2}\right)^n$, the series $\sum_{n=0}^{\infty} \left(\frac{-x^3}{2}\right)^n$ converges when $|\frac{-x^3}{2}| < 1$. Thus

$$|\frac{-x^3}{2}| < 1 \iff |x^3| < 2 \iff |x| < \sqrt[3]{2}$$

This shows that the radius of convergence is $R = \sqrt[3]{2}$. Moreover, the interval of convergence is $(-\sqrt[3]{2}, \sqrt[3]{2})$.

Expanding the power series representation for $f(x)$, we get that

$$\begin{aligned} f(x) &= \frac{x^3}{2} \sum_{n=0}^{\infty} \left(\frac{-x^3}{2}\right)^n \\ &= \frac{x^3}{2} \left(1 - \frac{x^3}{2} + \frac{x^6}{4} - \frac{x^9}{8} + \frac{x^{12}}{16} - \cdots \right. \\ &= \frac{x^3}{2} - \frac{x^6}{4} + \frac{x^9}{8} - \frac{x^{12}}{16} + \frac{x^{15}}{32} - \cdots \end{aligned}$$

We can now differentiate both sides to get

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{x^3}{2} - \frac{x^6}{4} + \frac{x^9}{8} - \frac{x^{12}}{16} + \frac{x^{15}}{32} - \cdots \right) \\ &= \frac{3x^2}{2} - \frac{3x^5}{2} + \frac{9x^8}{8} - \frac{12x^{11}}{16} + \frac{15x^{14}}{32} - \cdots \end{aligned}$$

We can also integrate both sides to get

$$\begin{aligned} \int f(x)\ dx &= \int \left(\frac{x^3}{2} - \frac{x^6}{4} + \frac{x^9}{8} - \frac{x^{12}}{16} + \frac{x^{15}}{32} - \cdots \right) \\ &= C + \frac{x^4}{8} - \frac{x^7}{28} + \frac{x^{10}}{80} - \frac{x^{13}}{208} + \frac{x^{16}}{512} - \cdots \end{aligned}$$

9. Consider the power series below.

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{n4^n}$$

Find the radius and interval of convergence for the power series.

Solution: We use the Ratio Test to obtain the radius and interval of convergence of this power series. Thus

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+1)4^{n+1}}\cdot\frac{n4^n}{(x+2)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}n4^n}{(x+2)^n(n+1)4^{n+1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(x+2)}{4}\cdot\frac{n}{n+1}\right| \\
&= \lim_{n\to\infty}\left|\frac{x+2}{4}\right|\lim_{n\to\infty}\frac{n}{n+1} \\
&= \left|\frac{x+2}{4}\right|(1) \\
&= \left|\frac{x+2}{4}\right|
\end{aligned}$$

Our series will converge when $|\frac{x+2}{4}| < 1$. Thus

$$\left|\frac{x+2}{4}\right| < 1 \iff |x+2| < 4 \iff -4 < x+2 < 4 \iff -6 < x < 2$$

It follows that our radius of convergence is $R = 4$. Moreover, the series converges when $x \in (-6, 2)$ and diverges when $x \in (-\infty, -6) \cup (2, \infty)$. However, we need to test the series for convergence or divergence when $x = -6$ and $x = 2$.

Suppose $x = -6$. Then our series becomes $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{n4^n}$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{2^n}{n4^n} &= \sum_{n=0}^{\infty}\frac{1}{n}\left(\frac{2}{4}\right)^n \\
&= \sum_{n=0}^{\infty}\frac{1}{n}\left(\frac{1}{2}\right)^n
\end{aligned}$$

We can compare this series to the geometric series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$, which converges since $|r| = |\frac{1}{2}| < 1$. Since $\displaystyle\frac{1}{n}\left(\frac{1}{2}\right)^n < \left(\frac{1}{2}\right)^n$ for all n, $\sum_{n=0}^{\infty}\frac{1}{n}\left(\frac{1}{2}\right)^n$ must converge by the Comparison Test. It follows that $x = -6$ is in the domain.

Suppose $x = 2$. Then our series becomes $\displaystyle\sum_{n=0}^{\infty}\frac{4^n}{n4^n} = \sum_{n=0}^{\infty}\frac{1}{n}$. This is the Harmonic Series which diverges. It follows that $x = 2$ is not in the domain.

We can conclude that our interval of convergence is $[-6, 2)$.

10. Let $g(x) = \cosh x = \frac{1}{2}\left(e^x - e^{-x}\right)$.
    (a) Find a Maclaurin series for $g(x)$. Write out at least the first four nonzero terms of the series. Then write your final conclusion in sigma notation.
    (b) What is the interval of convergence for the power series you found in part (a)?
    (c) Find a Taylor series for $g(x)$ centered at $a = \ln 2$. Write out at least the first four nonzero terms of the series. Do *not* write your final conclusion in sigma notation.
    (d) What is the interval of convergence for the power series you found in part (c)?
    (e) Find the third-degree polynomial $T_3(x)$ for $g(x)$ centered at $a = \ln 2$.

Solution: To find a Taylor or Maclaurin series, we need $f(x)$, $f'(x)$, $f''(x)$, $\ldots$, $f^{(n)}(x)$, $\ldots$. For the Maclaurin series, we will need to evaluate each derivative at $a = 0$. For the Taylor series in part (c), we will need to evaluate each derivative at $a = \ln 2$.

$f^{(n)}(x) =$	$a = 0$	$a = \ln 2$
$f(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$	$f(0) = \frac{1}{2}\left(e^0 - e^{-0}\right) = 0$	$f(\ln 2) = \frac{1}{2}\left(e^{\ln 2} - e^{-\ln 2}\right) = \frac{1}{2}\left(2 - \frac{1}{2}\right) = \frac{3}{4}$
$f'(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$	$f'(0) = \frac{1}{2}\left(e^0 + e^{-0}\right) = 1$	$f'(\ln 2) = \frac{1}{2}\left(e^{\ln 2} + e^{-\ln 2}\right) = \frac{1}{2}\left(2 + \frac{1}{2}\right) = \frac{5}{4}$
$f''(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$	$f''(0) = \frac{1}{2}\left(e^0 - e^{-0}\right) = 0$	$f''(\ln 2) = \frac{1}{2}\left(e^{\ln 2} - e^{-\ln 2}\right) = \frac{1}{2}\left(2 - \frac{1}{2}\right) = \frac{3}{4}$
$f'''(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$	$f'''(0) = \frac{1}{2}\left(e^0 + e^{-0}\right) = 1$	$f'''(\ln 2) = \frac{1}{2}\left(e^{\ln 2} + e^{-\ln 2}\right) = \frac{1}{2}\left(2 + \frac{1}{2}\right) = \frac{5}{4}$
$f^{(4)}(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$	$f^{(4)}(0) = \frac{1}{2}\left(e^0 - e^{-0}\right) = 0$	$f^{(4)}(\ln 2) = \frac{1}{2}\left(e^{\ln 2} - e^{-\ln 2}\right) = \frac{1}{2}\left(2 - \frac{1}{2}\right) = \frac{3}{4}$
$\vdots$		

(a) Now, for a Maclaurin series,

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
&= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\
&= 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \\
&= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}
\end{aligned}$$

(b) Using the Ratio Test, we can find a radius of convergence.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{x^{2n+1}}\right| &= \lim_{n\to\infty} \left|\frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}}\right| \\
&= \lim_{n\to\infty} \left|x^2 \cdot \frac{(2n+1)!}{(2n+3)!}\right| \\
&= x^2 \lim_{n\to\infty} \left|\frac{(2n+1)!}{(2n+3)!}\right| \\
&= x^2 \lim_{n\to\infty} \left|\frac{1}{(2n+3)(2n+2)}\right| \\
&= x^2 \cdot 0 \\
&= 0
\end{aligned}$$

This converges when $0 < 1$, so the radius of convergence is $R = \infty$. Thus, the interval of convergence is $\mathbb{R}$.

(c) Now, for a Taylor series centered at $x = a$,

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\
&= f(\ln 2) + \frac{f'(\ln 2)}{1!}(x - \ln 2) + \frac{f''(\ln 2)}{2!}(x - \ln 2)^2 + \frac{f'''(\ln 2)}{3!}(x - \ln 2)^3 + \cdots \\
&= \frac{3}{4} + \frac{\frac{5}{4}}{1!}(x - \ln 2) + \frac{\frac{3}{4}}{2!}(x - \ln 2)^2 + \frac{\frac{5}{4}}{3!}(x - \ln 2)^3 + \cdots \\
&= \frac{3}{4} + \frac{5}{4}(x - \ln 2) + \frac{3}{8}(x - \ln 2)^2 + \frac{5}{24}(x - \ln 2)^3 + \cdots
\end{aligned}$$

(d) By a parallel argument to (b), the interval of convergence will be $\mathbb{R}$.

(e) The 3rd-degree Taylor polynomial $T_3(x)$ centered at $a = \ln 2$ is

$$T_3(x) = \frac{3}{4} + \frac{5}{4}(x - \ln 2) + \frac{3}{8}(x - \ln 2)^2 + \frac{5}{24}(x - \ln 2)^3$$

11. Let $f(x) = \sqrt[5]{(1+x)^4}$.
    (a) Use the Binomial Series to expand $f(x)$ as a power series. Write the first four terms of the series in the expanded form as well as the summation notation.
    (b) What is the radius of convergence for the power series you found in part (a)?

Solution: Notice $f(x) = (1+x)^{\frac{4}{5}}$. By the Binomial Series,

$$\begin{aligned}
(1+x)^{\frac{4}{5}} &= \sum_{n=0}^{\infty} \binom{\frac{4}{5}}{n} x^n \\
&= 1 + \frac{4}{5}x + \frac{\left(\frac{4}{5}\right)\left(\frac{-1}{5}\right)}{2!}x^2 + \frac{\left(\frac{4}{5}\right)\left(\frac{-1}{5}\right)\left(\frac{-6}{5}\right)}{3!}x^3 + \frac{\left(\frac{4}{5}\right)\left(\frac{-1}{5}\right)\left(\frac{-6}{5}\right)\left(\frac{-11}{5}\right)}{4!}x^4 + \cdots \\
&\quad + \frac{\left(\frac{4}{5}\right)\left(\frac{-1}{5}\right)\cdots\left(\frac{4}{5} - n + 1\right)}{n!}x^n + \cdots \\
&= 1 + \frac{4}{5}x - \frac{2}{25}x^2 + \frac{4}{125}x^3 - \frac{11}{625}x^4 + \cdots
\end{aligned}$$

From the Binomial Series, we know that the radius of convergence is $R = 1$.

12. Let $g(x) = \dfrac{2}{(1+x)^4}$.
    (a) Use the Binomial Series to expand $g(x)$ as a power series. Write the first four terms of the series in the expanded form as well as the summation notation.
    (b) What is the radius of convergence for the power series you found in part (a)?

Solution: Notice $g(x) = 2(1+x)^{-4}$. So we need to expand $(1+x)^{-4}$. By the Binomial Series,

$$\begin{aligned}
(1+x)^{-4} &= \sum_{n=0}^{\infty} \binom{-4}{n} x^n \\
&= 1 - 4x + \frac{(-4)(-5)}{2!}x^2 + \frac{(-4)(-5)(-6)}{3!}x^3 + \frac{(-4)(-5)(-6)(-7)}{4!}x^4 + \cdots \\
&\quad + \frac{(-4)(-5)\cdots(-4-n+1)}{n!}x^n + \cdots \\
&= 1 - 4x + 20x^2 - 60x^3 + 140x^4 - \cdots + \frac{(-4)(-5)\cdots(-4-n+1)}{n!}x^n + \cdots
\end{aligned}$$

From the Binomial Series, we know that the radius of convergence is $R = 1$. It follows that

$$\begin{aligned} g(x) &= 2(1+x)^{-4} \\ &= \sum_{n=0}^{\infty} \binom{-4}{n} 2x^n \\ &= 2 - 8x + 40x^2 - 120x^3 + 280x^4 - \cdots + \frac{2(-4)(-5)\cdots(-4-n+1)}{n!} x^n + \cdots \end{aligned}$$

has radius of convergence $R = 1$.