13.10 Applications of Vector Calculus

13.10.1 Flow

Suppose a fluid with constant density ρ is flowing in a three-dimensional space, and $\mathbf{v}(x, y, z)$ is the velocity field for that fluid. Then $\mathbf{F} = \rho\mathbf{v}$ is the rate of flow per unit area of the fluid. Suppose further that $P_0(x_0, y_0, z_0)$ is a point in the fluid, and B_a is a ball around P_0 with radius $a > 0$.

We can approximate the divergence of $\mathbf{F}$ at any point $P \in B_a$ by using $\operatorname{div}\mathbf{F}(P) \approx \operatorname{div}\mathbf{F}(P_0)$, and we can use this to approximate the flux over the boundary sphere S_a of B_a as such.

$$\begin{aligned}
\iint\limits_{S_a} \mathbf{F} \cdot d\mathbf{S} &= \iiint\limits_{B_a} \operatorname{div}\mathbf{F}\ dV \\
&\approx \iiint\limits_{B_a} \operatorname{div}\mathbf{F}(P_0)\ dV \\
&= \operatorname{div}\mathbf{F}(P_0) \iiint\limits_{B_a} 1\ dV \\
&= \operatorname{div}\mathbf{F}(P_0) V(B_a)
\end{aligned}$$

Notice that as $a \to 0$, this approximation improves. Thus,

$$\operatorname{div}\mathbf{F}(P_0) = \lim_{a\to 0} \frac{1}{V(B_a)} \iint\limits_{S_a} \mathbf{F} \cdot d\mathbf{S}$$

Definition

Suppose $\mathbf{F}$ is the rate of flow per unit area at a point P. If $\operatorname{div}\mathbf{F}(P) > 0$, we say that P is a **source**. If $\operatorname{div}\mathbf{F}(P) < 0$, we say that P is a **sink**.

Example 1. Let $\mathbf{F} = \langle \ln|x| - \cos x, \sqrt{y} - y\sin y \rangle$. Let $P_1(2, 2)$ and $P_2(-2, 2)$ be points in the plane. Classify each of P_1 and P_2 as either a sink or a source.

13.10.2 Mass

Definition

A lamina is a 2-dimensional closed surface with mass and surface density.

Definition

If a lamina has the shape of a surface S and density (mass per unit area) $\rho(x, y, z)$, then the **mass** of the sheet is

$$m = \iint_S \rho(x, y, z)\ dS$$

and the **center of mass** is $(\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{1}{m} \iint_S x\rho(x, y, z)\ dS$$
$$\overline{y} = \frac{1}{m} \iint_S y\rho(x, y, z)\ dS$$
$$\overline{z} = \frac{1}{m} \iint_S z\rho(x, y, z)\ dS$$

Example 2. Find the mass and center of mass of the thin conical funnel $z = \sqrt{x^2 + y^2}$, $1 \leq z \leq 4$, if its density is $\rho(x, y, z) = 10 - z$.

13.10.3 Applications of Flux to Physics

Definition

If $\mathbf{E}$ is an electric field, then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the **electric flux** of $\mathbf{E}$ through the surface S.

Gauss' Law

Suppose $\mathbf{E}$ is an electric field. The net charge enclosed by a closed surface S is

$$Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}$$

Note: The number ε_0 is a constant called the **permittivity of free space** (or the **electric constant**) and is approximated by $\varepsilon_0 \approx 8.8542 \times 10^{-12}$ and is measured in $\frac{F}{m} = \frac{C^2}{Nm}$, where F is a Farad, C is a Coulomb, and N is a Newton.

Example 3. Suppose $\mathbf{E} = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ is an electric field. Find the net charge of $\mathbf{E}$ enclosed by the unit sphere. (Hint: See the section on Flux)

Definition

Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$. Then the **heat flow** is the vector field

$$\mathbf{F} = -K\nabla u$$

The constant K is defined experimentally and is called the **conductivity** of the substance.
The rate of heat flow across the surface S in the body is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}$$

Example 4. The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a at the center of the ball.