13.5 Curl & Divergence

13.5.1 Curl

Definition

We define the symbol ∇, pronounced "del" or "nabla", as the vector differential operator that operates on a multivariable function and outputs the vector field with partial derivatives of the function listed in corresponding components. That is,

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

And therefore

$$\begin{aligned}\nabla f &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f \\ &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \\ &= f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k} \\ &= \langle f_x, f_y, f_z \rangle\end{aligned}$$

Definition

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, R all exist, then we define the **curl** of $\mathbf{F}$ to be the vector field on $\mathbb{R}^3$ defined by

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k} \\ &= \nabla \times \mathbf{F}\end{aligned}$$

Note: The curl of a vector field is a definition of differentiation on a vector field that produces a vector field.

Note: The word "curl" is used because the curl has to do with rotating a vector field.

Example 1. Find the curl of $\mathbf{F}(x, y, z) = \ln(2y + 3z)\mathbf{i} + \ln(x + 3z)\mathbf{j} + \ln(x + 2y)\mathbf{k}$

Theorem

If f is a function of three variables with continuous second-order partial derivatives, then

$$\text{curl}(\nabla f) = \mathbf{0}$$

Proof:

The previous theorem can be restated as such:

Theorem

If $\mathbf{F}$ is conservative, then $\operatorname{curl}\mathbf{F} = \mathbf{0}$.

Note: The negation of this theorem states "If $\operatorname{curl}\mathbf{F} \neq \mathbf{0}$, then $\mathbf{F}$ is not conservative.

Example 2. Determine if $\mathbf{F}(x, y, z) = \ln(2y+3z)\mathbf{i}+\ln(x+3z)\mathbf{j}+\ln(x+2y)\mathbf{k}$ is conservative or not.

Theorem

If $\mathbf{F}$ is a vector field defined on $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\operatorname{curl}\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is conservative.

Definition

If $\mathbf{F}$ represents the velocity field for fluid flow and $\operatorname{curl}\mathbf{F} = \mathbf{0}$ at a point P, then the fluid is free from rotations at P, and we call $\mathbf{F}$ **irrotational** at P.

13.5.2 Divergence

Definition

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$, and P_x, Q_y, R_z exist, then we define the **divergence** of $\mathbf{F}$ to be the multivariable function (scalar field) defined by

$$\begin{aligned} \operatorname{div}\mathbf{F} &= P_x + Q_y + R_z \\ &= \nabla \cdot \mathbf{F} \end{aligned}$$

Note: The divergence of a vector field is a definition of differentiation on a vector field that produces a scalar field.

Example 3. If $\mathbf{F}(x, y, z) = \ln(2y + 3z)\mathbf{i} + \ln(x + 3z)\mathbf{j} + \ln(x + 2y)\mathbf{k}$, find $\operatorname{div}\mathbf{F}$.

Example 4. If $\mathbf{F}(x, y, z) = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}$, find $\operatorname{div}\mathbf{F}$.

Theorem

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$, and P, Q, R have continuous second-order partial derivatives, then

$$\operatorname{div} \operatorname{curl} \mathbf{F} = 0$$

Proof:

Example 5. Show that $\mathbf{F}(x, y, z) = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}$ cannot be written as the curl of another vector field.

We now have a vast collection of differential operators. Another one comes from looking at the divergence of a gradient vector field. That is

$$\text{div}(\nabla f) = \nabla \cdot (\nabla f)$$

We will define this new quantity as ∇^2.

Definition

If f is a scalar field with continuous second-order partial derivatives, then ∇^2 is called the **Laplace operator**, and **Laplace's equation** is defined as

$$\nabla^2 f = f_{xx} + f_{yy} + f_{zz}$$

Moreover, if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, then

$$\nabla^2 \mathbf{F} = \nabla^2 P\mathbf{i} + \nabla^2 Q\mathbf{j} + \nabla^2 R\mathbf{k}$$

Example 6. Let $f(x, y, z) = x^2yz + xy^2z + xyz^2$. Find $\nabla^2 f$.

Now that we have curl and divergence, we can revisit Green's Theorem with this new understanding.

Suppose D is a Cartesian region with boundary C and P, Q ass satisfy the requirements for Green's Theorem. Then for $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$,

$$\oint \mathbf{F} \cdot d\mathbf{r} = \oint_C P\ dx + Q\ dy$$

and if we consider $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + 0\mathbf{k}$, then $\mathbf{F}$ is now a vector field on $\mathbb{R}^3$, and

$$\begin{aligned} \text{curl}\,\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \end{aligned}$$

It follows that

$$\begin{aligned}(\operatorname{curl}\mathbf{F})\cdot\mathbf{k} &= \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathbf{k}\cdot\mathbf{k}\\ &= \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\end{aligned}$$

Green's Theorem in Vector Form

Let D be the Cartesian region bounded by a positively oriented, piecewise-smooth, simple closed curve C. If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ where P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint \mathbf{F}\cdot d\mathbf{r} = \oint_C P\ dx + Q\ dy = \iint_D (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\ dA$$

Since $d\mathbf{r} = \mathbf{T}\ ds$ This theorem expresses the line integral of the *tangential component* of $\mathbf{F}$ along C as the double integral of the vertical component of $\operatorname{curl}\mathbf{F}$ over D. What about the *normal component*?

Suppose C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ with $a \le t \le b$. Then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{j}$$

It can be shown that

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{j}$$

where $\mathbf{n}(t)$ is the outward normal vector to C. Then

$$\oint_C \mathbf{F}\cdot\mathbf{n}\ ds =$$

Theorem

$$\oint \mathbf{F}\cdot\mathbf{n}\ ds = \iint_D (\operatorname{div}\mathbf{F}(x,y))\ dA$$

That is, the line integral of the normal component of $\mathbf{F}$ along C is equal to the double integral of the divergence of $\mathbf{F}$ over D.