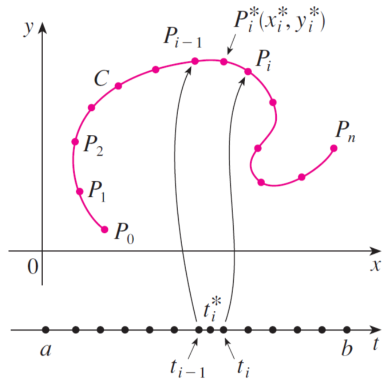
13.2 Line Integrals

Consider a plane curve C with parametric equations

$$x = x(t) \qquad y = y(t) \qquad a \le t \le b$$

and thus by the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = \langle x(t), y(t)\rangle$. For simplicity of computation, let's assume C is smooth.

Let's subdivide $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width $\Delta t = \frac{b-a}{n}$, and let $x_i = x(t_i)$ and $y_i = y(t_i)$, and define the corresponding point $P_i(x_i, y_i)$. Then C is divided into corresponding subarcs of length $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$, where Δs_i is the length of the part of the curve C between P_{i-1} and P_i. Choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc.



Now, if f is any function of two variables whose domain includes C, we evaluate f at (x_i^*, y_i^*) and multiply by Δs_i to obtain a type of Riemann sum – $\displaystyle\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$. Note that this is not a Riemann sum due to Δs_i not being constant, but it's close.

Definition

If f is defined on a smooth curve C whose equations are given by $x = x(t), y = y(t), a \le t \le b$, then the **line integral of f along C** is

$$\int_C f(x, y)\; ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\; \Delta s_i$$

provided this limit exists.

Now, we have previously found that the length of C is given by

$$L = \int_a^b ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

Since the Δs_i in question measure length of an arc, this formula is clearly important. In fact, if we make the substitution

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

we obtain

Theorem

If f is defined on a smooth curve C whose equations are given by $x = x(t), y = y(t), a \leq t \leq b$, then

$$\int_C f(x,y)\ ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Provided that the curve is traversed exactly once as t increases from a to b, this integral will always produce the same value regardless of the parametrization.

Example 1. Evaluate $\int_C (2 + x^2 y)\ ds$, where C is the upper half of the unit circle.

Definition

A curve C is called **piecewise-smooth** if C is a union of a finite number of smooth curves C_1, $C_2, \ldots, C_n$, where the terminal point of C_{i-1} is the initial point of C_i.

Definition

If C is a piecewise-smooth curve as described in the definition above, then

$$\int_C f(x,y)\ ds = \int_{C_1} f(x,y)\ ds + \int_{C_2} f(x,y)\ ds + \cdots + \int_{C_n} f(x,y)\ ds$$

Example 2. Evaluate $\displaystyle\int_C 2x\ ds$, where C is the curve that follows $y = x^2$ from the origin to $(1,1)$, then travels linearly from $(1,1)$ to $(1,2)$.

If you want to set up a line integral with respect to x or y, then that is also plausible. In fact,

Definition

If f is defined on a smooth curve C whose equations are given by $x = x(t), y = y(t), a \le t \le b$, then

$$\int_C f(x,y)\ dx = \int_a^b f(x(t),y(t))\ x'(t)\ dt$$
$$\int_C f(x,y)\ dy = \int_a^b f(x(t),y(t))\ y'(t)\ dt$$
$$\int_C f(x,y)\ ds = \int_a^b f(x(t),y(t))\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

where the first integral is a **line integral of f along C with respect to x**, the second is a **line integral of f along C with respect to y**, and the final (original) integral is a **line integral with respect to arc length**.

Note: When setting up a line integral, it is often difficult to come up with a parametric representation for C. It is often useful to consider the parametric representation for C whose initial point is given by $\mathbf{r}_0$ and terminal point is given by $\mathbf{r}_1$ as follows

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \le t \le 1$$

Example 3. Evaluate $\displaystyle\int_C y^2\,dx + x\,dy$ for two different curves C beginning at $(-5,-3)$ and ending at $(0,2)$.

a. C is the line segment from $(-5,-3)$ to $(0,2)$.

b. C is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

Make a generalization about based on your results.

A similar line integral can be defined for three-dimensional space. That is,

Definition

If f is defined on a smooth curve C whose equations are given by $x = x(t), y = y(t), z = z(t), a \leq t \leq b$, then

$$\int_C f(x,y,z)\ dx = \int_a^b f(x(t),y(t),z(t))\ x'(t)\ dt$$
$$\int_C f(x,y,z)\ dy = \int_a^b f(x(t),y(t),z(t))\ y'(t)\ dt$$
$$\int_C f(x,y,z)\ dy = \int_a^b f(x(t),y(t),z(t))\ z'(t)\ dt$$
$$\int_C f(x,y,z)\ ds = \int_a^b f(x(t),y(t),z(t))\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

Example 4. Evaluate $\int_C y \sin z\ ds$, where C is the circular helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ with $t \in [0, 2\pi]$. *Note that* $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$.

Definition

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$ with $a \le t \le b$. The **line integral of F along** C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt = \int_C \mathbf{F} \cdot \mathbf{T}\ ds$$

Note:

- $\mathbf{F}$ is an abbreviation of $\mathbf{F}(x, y, z)$
- $d\mathbf{r}$ is an abbreviation for $\mathbf{r}'(t)\ dt$
- $\mathbf{F}(\mathbf{r}(t))$ is an abbreviation for $\mathbf{F}(x(t), y(t), z(t))$

Notice that if $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$, then

$$\begin{aligned}\mathbf{F} \cdot d\mathbf{r} &= (P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}) \cdot (x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}) \\ &= P(x, y, z)x'(t)\mathbf{i} + Q(x, y, z)y'(t)\mathbf{j} + R(x, y, z)z'(t)\mathbf{k}\end{aligned}$$

Theorem

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a continuous vector field defined on a smooth curve C, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\ dx + Q\ dy + R\ dz$$