13.7 Stokes' Theorem

Stokes' Theorem is a generalization of Green's Theorem. Green's Theorem relates a planar double integral to a line integral over its boundary. Stokes' Theorem relates a surface integral to a line integral over its boundary. With Green's Theorem, we had to have the line integral traversed in a positive orientation. With Stokes' Theorem, we need the same thing, but we need to generalize that definition to work for not-necessarily-planar curves.

Definition

Suppose S is an oriented surface with boundary C. If C is traversed by a "body" in the positive direction with the body in the direction of the unit normal vectors of the surface, and this traversing results in the surface being on the left of the body, then we say that the orientation of S induces the **positive orientation of the boundary curve** C.

Stokes' Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. If $\mathbf{F}$ is a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

That is, the flux of the curl of $\mathbf{F}$ across the surface is the same as the line integral of $\mathbf{F}$ along the boundary, considering all of the assumptions (which is fairly standard).

Recall

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \qquad \text{and} \qquad \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$$

Stokes' Theorem says that the line integral around the boundary of S of the tangential component of $\mathbf{F}$ is the same as the surface integral over s of the normal component of the curl of $\mathbf{F}$.

Moreover, if all of the previous assumptions are true, and S is flat and in the xy-plane, then $\mathbf{n} = \mathbf{k}$, and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dS = \int_S \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA$$

Therefore, Green's Theorem is really just a special case of Stokes' Theorem.

The proof of a special case of Stokes' Theorem will be at the end of this section.

Example 1. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle -y^2, x, z^2 \rangle$ and C is the intersection of $y + z - 2 = 0$ and $x^2 + y^2 = 1$.

Example 2. Evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle xz, yz, xy \rangle$, and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ inside of $x^2 + y^2 = 1$ above the xy-plane.

Proof of a Special Case of Stokes' Theorem: