13.8 The Divergence Theorem

Definition

A surface that is simultaneous type 1, 2, and 3 (simple in the x, y, and z directions) is called a **simple solid region**.

Recall one version of Green's Theorem that states

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$

where C is a positively-oriented boundary curve of the Cartesian region D.

The Divergence Theorem

Let E be a simple solid region and S be the boundary surface of E with positive orientation. If $\mathbf{F}$ is a vector field whose component functions have continuous partial derivatives on an open region containing E, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

That is, the flux of $\mathbf{F}$ across S is the same as the triple integral of the divergence of $\mathbf{F}$ over E, considering all of the assumptions (which is fairly standard).

Proof:

Example 1. Find the flux of $\mathbf{F} = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ over the unit sphere.

Example 2. Find the flux of $\mathbf{F} = xy\mathbf{i} + \left(y^2 + e^{xz^2}\right)\mathbf{j} + \sin(xy)\mathbf{k}$ over the surface of the solid E bounded by the parabolic cylinder $x^2 + z = 1$, the xy-coordinate plane, the xz-coordinate plane, and the plane $y + z = 2$.