12.7 Triple Integrals

12.7.1 Triple Integrals over a Box

Previously,

- $\displaystyle\int_a^b f(x)\,dx$ integrates f over an interval – a subset of $\mathbb{R}$.
- $\displaystyle\iint_{\mathscr{D}} f(x,y)\,dA$ integrate f over a region – a subset of $\mathbb{R}^2$.

Via analogy (rather than exploration), we make the following definition.

Definition

The **triple integral of f over the box B** is

$$\iiint_{\mathscr{B}} f(x,y,z)\,dV = \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V$$

provided the limit exists.

Just as in double integrals, Fubini's Theorem applies to triple integrals.

Theorem

Fubini's Theorem: If f is continuous on the rectangular box $B = [a,b]\times[c,d]\times[r,s]$, then

$$\iiint_{\mathscr{B}} f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Note that the limits of integration are constants for x, y, and z.

Example 1. Evaluate the integral $\displaystyle\iiint_{\mathscr{B}} (xy+z^2)\,dV$, where $\mathscr{B} = \{(x,y,z) \mid 0 \le x \le 2, 0 \le y \le 1, 0 \le z \le 3\}$.

12.7.2 Triple Integrals over a General Solid

Just as in double integration, limits of integration do not necessarily need to be constant, and we use the same procedure as before to deal with such situations. The strategy to deal with such situations is dependent upon the solid of which we are integrating over. As long as the region E that we are integrating over is bounded, we can enclose E within a larger box B and define a new function

$$F(x,y,z) = \begin{cases} f(x,y,z) & ,(x,y,z) \in E \\ 0 & ,(x,y,z) \notin E \end{cases}$$

It follows that $\iiint\limits_{\mathcal{W}} f(x,y,z)\ dV = \iiint\limits_{\mathscr{B}} F(x,y,z)\ dV$. This integral will exist as long as the boundary on $\mathcal{W}$ is *reasonably smooth*, and this integral will have nearly all of the same properties as double integrals.

Definition

We define the **triple integral of f over a general solid** $\mathcal{W}$ to be $\iiint\limits_{\mathcal{W}} f(x,y,z)\ dV = \iiint\limits_{\mathscr{B}} F(x,y,z)\ dV$ where

$$F(x,y,z) = \begin{cases} f(x,y,z) & ,(x,y,z) \in \mathcal{W} \\ 0 & ,(x,y,z) \notin \mathcal{W} \end{cases}$$

provided the integral exists.

In order to evaluate a triple integral over a general solid, we observe what sort of a region we have. For subsets of $\mathbb{R}^2$, we have Type I and Type II regions. For subsets of $\mathbb{R}^3$, we have Type I, Type II, and Type III regions.

Definition

Let $\mathcal{W}$ be a solid region in $\mathbb{R}^3$, and let $\mathscr{D}$ be the projection of $\mathcal{W}$ onto the xy-plane.

- The solid $\mathcal{W}$ is **Type I solid region** if it lies between the graphs of two continuous functions of x and y. That is,

$$\mathcal{W} = \{(x, y, z) \mid (x, y) \in \mathscr{D}, u_1(x, y) \leq z \leq u_2(x, y)\}$$

- The solid $\mathcal{W}$ is **Type II solid region** if it lies between the graphs of two continuous functions of y and z. That is,

$$\mathcal{W} = \{(x, y, z) \mid (y, z) \in \mathscr{D}, u_1(y, z) \leq x \leq u_2(y, z)\}$$

- The solid $\mathcal{W}$ is **Type III solid region** if it lies between the graphs of two continuous functions of x and z. That is,

$$\mathcal{W} = \{(x, y, z) \mid (x, z) \in \mathscr{D}, u_1(x, z) \leq y \leq u_2(x, z)\}$$

The strategy for evaluating a double integral is to turn it into two single-variable integrals.

Strategy

The strategy for evaluating a triple integral is to turn it into a single-variable integral and a double integral. For example, if $\mathcal{W}$ is a Type I solid region, then

$$\iiint\limits_{\mathcal{W}} f(x, y, z)\ dV = \iint\limits_{\mathscr{D}} \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\ dz \right)\ dA$$

Upon integration, the inside integral will leave a function of x and y, leading to a double integral that we know how to handle.

Example 2. Evaluate $\displaystyle\iiint\limits_{\mathcal{W}} z\ dV$, where $\mathcal{W}$ is the region between the planes $z = x + y$ and $z = 3x + 5y$ lying over the rectangle $\mathscr{D} = [0, 3] \times [0, 2]$.

Example 3. Evaluate $\iiint\limits_{\mathcal{W}} z \, dV$, where $\mathcal{W}$ is the region between the planes $z = x + y$ and $z = 3x + 5y$ lying over the triangle on the xy-plane whose vertices are $(0, 0)$, $(1, 0)$, and $(0, 1)$.

12.7.3 Size and Average Values

As mentioned before, triple integrals inherit a lot of the same properties as double and single integrals. In particular, we have notions of size and average value.

Theorem

Integrating 1 will produce the size of the region of integration. That is,

- If integrating over an interval in $\mathbb{R}$, then

$$\text{size}([a,b]) = \int_a^b 1\ dx$$

  where the size is the length of the interval.

- If integrating over a planar region in $\mathbb{R}^2$, then

$$\text{size}(\mathscr{D}) = \iint_{\mathscr{D}} 1\ dA$$

  where the size is the area of the region.

- If integrating over a solid $\mathcal{W}$ in $\mathbb{R}^3$, then

$$\text{size}(\mathcal{W}) = \iiint_{\mathcal{W}} 1\ dV$$

  where the size is the volume of the solid.

Moreover, the average value of an integrable function f can be found by integrating over the region of integration and dividing by the size of that region. That is,

- If f is an integrable function of x, then

$$\overline{f}_{[a,b]} = \frac{1}{\text{size}\,[a,b]} \int_a^b f(x)\ dx$$

- If f is an integrable function of x and y, then

$$\overline{f}_{\mathscr{D}} = \frac{1}{\text{size}\,\mathscr{D}} \iint_{\mathscr{D}} f(x,y)\ dA$$

- If f is an integrable function of x, y, and z, then

$$\overline{f}_{\mathcal{W}} = \frac{1}{\text{size}\,\mathcal{W}} \iiint_{\mathcal{W}} f(x,y,z)\ dV$$

Example 4. A crystal $\mathcal{W}$ is pulled extracted out of a cold case and set on its side. A coordinate system is placed around the crystal so that it exists in the first octant bounded by the coordinate planes, $x + z = 1$, and $x + y + z = 3$. The temperature at every point in the crystal is given by $T(x, y, z) = x$, measured in $^\circ$ C.

a. Find the volume of the crystal.

b. Find the average temperature of the crystal.

Example 5. Find the value of the integral of $f(x, y, z) = x$ over the region $\mathcal{W}$ bounded above by $z = 4 - x^2 - y^2$ and below by $z = x^2 + 3y^2$ in the first octant.