Math 252
Final Review Key

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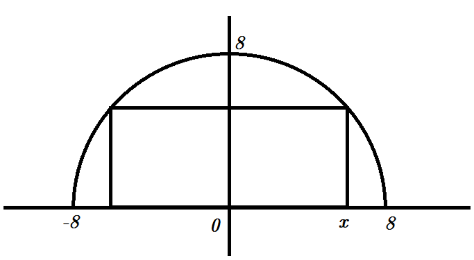
1 Conceptual Questions

1. What makes an integral an improper integral?
2. What kind of shape is used for each area approximation: Left- and right-endpoint, Trapezoidal Rule, Midpoint Rule, and Simpson's Rule?
3. When finding the volume of a solid of revolution, describe when a disk method is useful. Describe when a washer method is useful. Describe when a shell method is useful.
4. What is the relationship between S_n, M_n, and T_n?
5. When can we use L'Hôpital's Rule?
6. List the indeterminate forms.
7. Why can't we use the Fundamental Theorem of Calculus Part II to integrate $\int_1^3 \frac{1}{x-2}\,dx$?

2 Computational Questions

1. A rectangle has its base on the x-axis and its upper two vertices on the semicircle $y = \sqrt{64 - x^2}$. What is the largest area the rectangle can have. Use calculus and show all work in order to receive credit.

Solution:



$$\begin{aligned} A &= 2xy \\ A(x) &= 2x\sqrt{64 - x^2} \\ A'(x) &= 2\sqrt{64 - x^2} - \frac{2x^2}{\sqrt{64 - x^2}} \\ &= \frac{-4x^2 + 128}{\sqrt{64 - x^2}} \end{aligned}$$

Looking for the critical numbers, we solve $A'(x) = 0$ and find when $A'(x)$ is undefined. When $x = \pm 8$, A' is undefined, but we discount those cases, for we would not have a rectangle for those values.

$$
\begin{aligned}
0 = A'(x) &= \frac{-4x^2 + 128}{\sqrt{64 - x^2}} \\
0 &= -4x^2 + 128 \\
x^2 &= 32 \\
x &= \pm 4\sqrt{2}
\end{aligned}
$$

The plus and minus values of x make the same rectangle, so let's take $x = 4\sqrt{2}$. Then $y = \sqrt{64 - (4\sqrt{2})^2} = 4\sqrt{2}$, and $A = (8\sqrt{2})(4\sqrt{2}) = 64$. The largest area the rectangle can have is 64 square units.

2. An object moves along a line so that its velocity at time t is $v(t) = 3t^2 - 22t + 24$ meters per second. Find the displacement and total distance traveled by the object for $0 \leq t \leq 8$.

Solution: Displacement is $\int v(t)\, dt$, while distance is $\int |v(t)|\, dt$. Now, $v(t) \geq 0$ on $\left(-\infty, \frac{4}{3}\right) \cup (6, \infty)$. This information is useful for distance.

$$
\begin{aligned}
\text{Displacement} &= \int_0^8 (3t^2 - 22t + 24)\ dt \\
&= \left[t^3 - 11t^2 + 24t\right]_0^8 \\
&= \left[(8)^3 - 11(8)^2 + 24(8)\right] - \left[(0)^3 - 11(0)^2 + 24(0)\right] \\
&= 0 \text{ meters} \\
\text{Distance} &= \int_0^8 |3t^2 - 22t + 24|\ dt \\
&= \int_0^{\frac{4}{3}} (3t^2 - 22t + 24)\ dt - \int_{\frac{4}{3}}^6 (3t^2 - 22t + 24)\ dt + \int_6^8 (3t^2 - 22t + 24)\ dt \\
&= \left[t^3 - 11t^2 + 24t\right]_0^{\frac{4}{3}} - \left[t^3 - 11t^2 + 24t\right]_{\frac{4}{3}}^6 + \left[t^3 - 11t^2 + 24t\right]_6^8 \\
&= \frac{2744}{27} \text{ meters}
\end{aligned}
$$

3. Evaluate $\int_0^7 (x^4 - 8x + 7)\ dx$.

Solution:

$$\int_0^7 (x^4 - 8x + 7)\ dx = \left[\frac{x^5}{5} - 4x^2 + 7x\right]_0^7 = \frac{16072}{5}$$

4. Evaluate $\int_0^1 (1-r)^9\ dr$.

Solution: Let $u = 1 - r$. Then $du = -dr$, so

$$\int_0^1 (1-r)^9\ dr = -\int_0^1 (1-r)^9\ dr = -\int_1^0 u^9\ du = \int_0^1 u^9\ du = \left.\frac{u^{10}}{10}\right|_0^1 = \frac{1}{10}$$

5. Evaluate $\int \frac{9x^2}{\sqrt[3]{x^3+2}}\ dx$.

Solution: Let $u = x^3 + 2$. Then $du = 3x^2\ dx$, so

$$\begin{aligned}\int \frac{9x^2}{\sqrt[3]{x^3+2}}\ dx &= 3\int u^{-\frac{1}{3}}\ du \\ &= \frac{9}{2}u^{\frac{2}{3}} + C = \frac{9}{2}(x^3+2)^{\frac{2}{3}} + C\end{aligned}$$

6. Evaluate $\int \sin^3 x \cos x\ dx$.

Solution: Let $u = \sin x$. Then $du = \cos x\ dx$, so

$$\int \sin^3 x \cos x\ dx = \int u^3\ du = \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4 x + C$$

7. Evaluate $\int_{-1}^1 \cos x \tan x\ dx$.

Solution: Since $\cos x$ is even and $\tan x$ is odd, $\cos x \tan x$ is odd, so $\int_{-1}^1 \cos x \tan x\ dx = 0$.

8. Evaluate $\int \frac{6x}{\sqrt{3x^2-1}}\ dx$.

Solution: Let $u = 3x^2 - 1$. Then $du = 6x\ dx$, so

$$\int \frac{6x\ dx}{\sqrt{3x^2-1}} = \int u^{-\frac{1}{2}}\ du = 2\sqrt{u} + C = 2\sqrt{3x^2-1} + C$$

9. Evaluate $\displaystyle\int \frac{\ln(\ln x)}{x}\ dx$.

Solution: Let $u = \ln x$. Then $du = \frac{1}{x}\ dx$. Then

$$\int \frac{\ln(\ln x)}{x}\ dx = \int \ln u\ du = u \ln u - u + C = (\ln x)\ln(\ln x) - \ln x + C$$

10. Evaluate $\displaystyle\int x^2 e^{-x}\ dx$.

Solution: Integration by parts. Let

$$\begin{array}{ll} u = x^2 & dv = e^{-x}\ dx \\ du = 2x\ dx & v = -e^{-x} \end{array}$$

$$\int x^2 e^{-x}\ dx = -x^2 e^{-x} + \int 2x e^{-x}\ dx = -x^2 e^{x} + 2\int x e^{-x}\ dx$$

By parts again, we get

$$\begin{array}{ll} u = x & dv = e^{-x}\ dx \\ du = dx & v = -e^{-x} \end{array}$$

$$\int x e^{-x}\ dx = -x e^{-x} + \int e^{-x}\ dx = -x e^{-x} - e^{-x}$$

It follows that

$$\int x^2 e^{-x}\ dx = -x^2 e^{-x} + 2\int x e^{-x}\ dx = -x^2 e^{-x} - 2(x e^{-x} + e^{-x}) + C = -e^{-x}(x^2 + 2x + 2) + C$$

11. Evaluate $\displaystyle\int \frac{4x+1}{x(x^2-4)}\ dx$.

Note: This question contains a denominator that is more complicated than one you will be assessed on. However, it is a doable problem, so I leave it on here to try.

Solution: No cancelation or substitution is useful, so we proceed by partial fractions.

$$\frac{4x+1}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

By the Heaviside Coverup (or multiply by the LCD and equate coefficients),

$$\frac{4x+1}{x(x-2)(x+2)} = \frac{-\frac{1}{4}}{x} + \frac{\frac{9}{8}}{x-2} - \frac{\frac{7}{8}}{x+2}$$

So then we can integrate

$$\begin{aligned} \int \frac{4x+1}{x(x-2)(x+2)}\ dx &= \int \left[\frac{-\frac{1}{4}}{x} + \frac{\frac{9}{8}}{x-2} - \frac{\frac{7}{8}}{x+2}\right]\ dx \\ &= -\frac{1}{4}\ln|x| + \frac{9}{8}\ln|x-2| - \frac{7}{8}\ln|x+2| + C \end{aligned}$$

12. Evaluate $\int \cos^3 x \, dx$.

Solution:

$$\begin{aligned}\int \cos^3 x \, dx &= \int \cos^2 x \cos x \, dx \\ &= \int (1 - \sin^2 x) \cos x \, dx \\ &= \int \cos x \, dx - \int \sin^2 x \cos x \, dx \\ &= \sin x - \frac{1}{3} \sin^3 x + C\end{aligned}$$

13. Evaluate $\int_4^6 \frac{2}{5-x} \, dx$. If the integral diverges, show the work that leads to your conclusion.

Solution: Since $\frac{2}{5-x}$ has an infinite discontinuity at $x = 5$, this integral must be split into two improper integrals as such.

$$\begin{aligned}\int_4^6 \frac{2}{5-x} \, dx &= \lim_{t \to 5^-} \int_4^t \frac{2}{5-x} \, dx + \lim_{t \to 5^+} \int_t^6 \frac{2}{5-x} \, dx \\ &= \lim_{t \to 5^-} -2 \ln|5-x| \Big|_4^t + \lim_{t \to 5^+} -2 \ln|5-x| \Big|_t^6 \\ &= -2 \left(\lim_{t \to 5^-} (\ln|5-t| - \ln 1) + \lim_{t \to 5^+} (\ln 1 - \ln|5-t|) \right)\end{aligned}$$

Since $\lim_{t \to 5^-} \ln|5-t| = -\infty$, our original limit does not exist. Therefore, the integral diverges.

14. Find the area of the region bounded by the curves $y = \frac{1}{1+x^2}$, $y = 1 + \ln(x+1)$, $x = 0$, and $x = 1$.

Solution: Note $1 + \ln(x+1) \geq \frac{1}{1+x^2}$ for all $0 \leq x \leq 1$. So the area between the curves is given by

$$\begin{aligned}\int_0^1 \left[(1 + \ln(x+1)) - \frac{1}{1+x^2} \right] dx &= [x + (x+1)\ln(x+1) - (x+1) - \arctan x]_0^1 \\ &= (1 + 2\ln 2 - 2 - \arctan 1) - (0 + 1 \ln 1 - 1 - \arctan 0) \\ &= -1 + 2\ln 2 - \frac{\pi}{4} + 1 = 2\ln 2 - \frac{\pi}{4}\end{aligned}$$

15. Find the volume of the solid obtained by rotating the region bounded by $y = 2x$ and $y = x^2$ about the y-axis.

Solution: The solid will have horizontal cross sections being a washer, so $V = \int_a^b \pi(R_2^2 - R_1^2)\ dy$. Since we are integrating with respect to y, we need the curves to be $x = f(y)$, so

$$y = 2x \Longrightarrow x = \frac{1}{2}y \qquad\qquad y = x^2 \Longrightarrow x = \sqrt{y}$$

Moreover, we need the limits of integration, so

$$\frac{1}{2}y = \sqrt{y} \Longrightarrow y = 0, 4$$

Thus,

$$\begin{aligned} V &= \int_0^4 \pi \left[(\sqrt{y})^2 - \left(\frac{1}{2}y\right)^2 \right]\ dy \\ &= \pi \int_0^4 \left(y - \frac{1}{4}y^2 \right)\ dy \\ &= \pi \left[\frac{1}{2}y^2 - \frac{1}{12}y^3 \right]_0^4 \\ &= \pi \left(8 - \frac{16}{3} \right) = \frac{8}{3}\pi \end{aligned}$$

16. Find the volume of the solid obtained by rotating the region bounded by $y = xe^{-x}$, $0 \le x \le 2$, about the y-axis.

Solution: The solid will have horizontal cross sections with both radii coming from the same curve, so we need to use the shell method, so $V = \int_a^b 2\pi rh\ dx$. The radius of each shell is $r = x$, and the height is $h = xe^{-x}$. Thus,

$$\begin{aligned} V &= \int_0^2 2\pi x \left(xe^{-x} \right)\ dx \\ &= 2\pi \int_0^2 x^2 e^{-x}\ dx \qquad \text{Integration by Parts} \\ &= 2\pi e^{-x}(-x^2 - 2x - 2)\Big|_0^2 \\ &= 2\pi \left(2 - 10e^{-2} \right) \\ &= 4\pi \left(1 - 5e^{-2} \right) \end{aligned}$$

17. Find an integral that represents the length of the curve $y = \ln|\sec x|$, $0 \le x \le \frac{\pi}{4}$.

Solution:

$$L = \int ds = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = \ln|\sec x|$$

$$\frac{dy}{dx} = \frac{1}{\sec x} \sec x \tan x = \tan x$$

$$\left(\frac{dy}{dx}\right)^2 = \tan^2 x$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} |\sec x|\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec x\, dx \qquad \text{since } \sec x \ge 0 \text{ for } 0 \le x \le \frac{\pi}{4}$$

18. Damien used to be a runner (not a very good one, mind you). His friend took speed readings in m/s for the second and recorded the results as follows

t	0	1	2	3	4	5	6	7	8	9	10
$v(t)$	0	8.2	12.6	14.0	16.1	18.5	18.9	20.2	20.1	16.1	3.2

Use Simpson's Rule to estimate how far Damien sprinted before his legs gave out.

Solution: We begin by removing judgment from the context of the problem and strictly looking at the data and interpreting its meaning. ☺

$$S = \frac{\Delta x}{3}(v(0) + 4v(1) + 2v(2) + 4v(3) + 2v(4) + 4v(5) + 2v(6) + 4v(7) + 2v(8)$$
$$+ 4v(9) + v(10))$$
$$= \frac{1}{3}(0 + 4(8.2) + 2(12.6) + 4(14.0) + 2(16.1) + 4(18.5) + 2(18.9) + 4(20.2) + 2(20.1)$$
$$+ 4(16.1) + 3.2)$$
$$= \frac{1}{3}(446.6) \approx 148.87$$

Thus, Damien ran about 148.87 meters before hitting the ground.