SEPARABLE EQUATIONS

MTH 253 LECTURE NOTES

So far, we have verified solutions to differential equations and have graphed solutions to differential equations using direction fields. You might be wondering "How do we actually solve differential equations?" That is, how do we find an explicit algebraic function that is the solution (or family of solutions) to a differential equation? We will now focus on a specific class of differential equations in which we can find algebraic formulas for their solutions; these are called separable equations.

Definition

A **separable equation** is any equation that can be written in the form

$$\frac{dy}{dx} = f(x)g(y) \qquad \text{or} \qquad \frac{dy}{dx} = \frac{f(x)}{g(y)}$$

Note: This algebraic structure allows the variables x and y to be *separated* onto opposite sides of the equation, and then we can find an algebraic formula for the solution by using integration. This strategy is called *separation of variables* and is introduced here by treating dx and dy as differentials. We present this well-established strategy without discussing *why* this works.

Note: Separable equations must involve multiplication or division of two functions $f(x)$ and $g(y)$, and cannot involve addition or subtraction. Consider $\frac{dy}{dx} = x + y$.

Example 1. Consider the differential equation $\frac{dy}{dx} = (x+1)e^y$.

a. Solve the differential equation using separation of variables.
b. Find the solution curve that passes through the origin.
c. Check your answers to (a) and (b) in GeoGebra using the `slopefield` command and graphing the solutions.

Exercise 1. Consider the differential equation $\frac{dy}{dx} = \frac{x^2}{1-y^2}$.

a. Solve the differential equation using separation of variables. You may leave the solution in implicit form (that is, do not solve explicitly for y).
b. Find the solution that satisfies the initial condition $y(-1) = 0$.
c. Check your answers to (a) and (b) in GeoGebra using the `slopefield` command and graphing the solutions.

Example 2. Solve the differential equation $\dfrac{dy}{dx} = x\cos(x)y\ln(y)$. Be sure to also state any equilibrium solutions.

Example 3. Find the family of functions whose slope at (x, y) is $\frac{y}{x}$. Reflect on why this particular family of functions makes sense as solutions to this differential equation.