SEQUENCES

MTH 253 LECTURE NOTES

Definition:

A **Sequence** is a list of numbers written in a definite order. There are a couple ways to notate this:

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty} \qquad \text{or} \qquad \{a_n\}$$

where a_1 is the **first term**, a_2 is the **second term**, a_n is the nth term, and so on.

Example 1. Consider the sequence of terms a_n below.

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\right\}$$

a. What is a_{27}?

b. What is a_n?

c. Find the general form for the sequence and write it in the $\{a_n\}_{n=k}^{\infty}$ notation.

Example 2. Find the general form of the following sequences and write it in the $\{a_n\}_{n=k}^{\infty}$ notation where k is the start of the sequence. Answers may vary!

a. $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots\right\}$

b. $\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\right\}$

c. $\{-1, 1, -1, 1, -1, 1, ...\}$

d. $\{1, 3, 5, 7, 9, ...\}$

e. $\{1, -8, 27, -64, 125, ...\}$

f. $\{0, 1, 1, 2, 3, 5, 8, 13, 21, ...\}$

Exercise 1. Find the general form of the following sequences and write it in the $\{a_n\}_{n=k}^{\infty}$ notation where k is the start of the sequence.

a. $\left\{2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25}, ...\right\}$

b. $\left\{-1, \frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, -\frac{1}{81}, ...\right\}$

Definition:

The notation $\lim_{n\to\infty} a_n = L$ means that the terms of $\{a_n\}$ can be made as close to L as we like by taking n to be sufficiently large. If the limit exists, we say that the sequence **converges**. Otherwise, the sequence **diverges**.

Note: The most important goal for us at this point is to determine whether a sequence converges or diverges. Beyond that, if we have a convergent sequence, it is often important to know what it converges to. We will establish and recover a lot of theory in order to determine convergence or divergence.

Example 3. Determine whether each sequence in Example 2 converges or diverges. If it converges, find the limit.

Exploration: There are two primary ways to graph a sequence

- A number line
- A Cartesian plane (Horizontal n-axis, Vertical a_n-axis)

Example 4. Graph $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ using each of the methods below. Then, determine whether the sequence converges or diverges.

a. A number line.

b. A Cartesian plane

Exercise 2. Graph the sequence $\left\{\dfrac{1}{3^n}\right\}_{n=1}^{\infty}$ on a Cartesian plane. Then determine whether the sequence converges or diverges.

Definition:

A sequence $\{a_n\}$ is called a **Geometric Sequence** if each term of the sequence is obtained from the previous term by multiplying by a **common ratio** r. Often, we write

$$a_n = ar^{n-1}$$

where a is the **initial term** of the sequence.
Note that the exponent of $n-1$ may change a bit! The important part is that each term is obtained from the previous by multiplying by r.

Theorem:

The sequence $\{r^n\}$ is convergent if $r \in (-1, 1]$ and divergent otherwise. Moreover

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Note: This means that a geometric sequence is convergent if and only if it has a common ratio $r \in (-1, 1]$.

Technology Exploration: https://www.desmos.com/calculator/msjqj5tnrm

Context: Below is a bunch of theory of limits of continuous functions that we can recover from Differential Calculus. We will be able to utilize this theory in order to determine whether a sequence converges or diverges.

Theorem:

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is a positive integer, then $\lim_{n\to\infty} a_n = L$.

Exploration:

From Differential Calculus, $\lim_{x\to\infty} \frac{1}{x^r} = 0$ when $r > 0$, so

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \qquad \text{whenever } r > 0$$

Moreover, just as before, if $a_n \longrightarrow \infty$ as $n \longrightarrow \infty$, then we say $\lim_{n\to\infty} a_n = \infty$.

Theorem:

Limit Laws for Sequences: If $\{a_n\}$ and $\{b_n\}$ are convergent sequences, then

1. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
3. $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$
4. $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \qquad \text{if } \lim_{n\to\infty} b_n \neq 0$
6. $\lim_{n\to\infty}(a_n)^p = \left[\lim_{n\to\infty} a_n\right]^p \qquad p > 0, a_n > 0$

Theorem:

Squeeze Theorem for Sequences: If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem:

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Proof: Notice $\lim_{n\to\infty}(-|a_n|) = -\lim_{n\to\infty} a_n = 0$. Since $-|a_n| \leq a_n \leq |a_n|$, and since $\lim_{n\to\infty} |a_n| = 0$, we can conclude that $\lim_{n\to\infty} a_n = 0$ by the Squeeze Theorem.

□

Theorem:

If $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Example 5. Determine whether the following sequences converge or diverge. If the sequence converges, determine its limit. Justify your conclusion as specifically as possible.

a. $\{(-1)^n\}$

b. $\left\{\frac{(-1)^n}{n}\right\}$

c. $\left\{\sin\frac{\pi}{n}\right\}$

d. $\left\{\frac{n!}{n^n}\right\}$

Exercise 3. Determine whether the following sequences converge or diverge. If the sequence converges, determine its limit. Justify your conclusion as specifically as possible.

a. $\left\{\frac{n^3}{n^3+1}\right\}$

b. $\left\{\frac{\ln n}{n}\right\}$

c. $\{\arctan(2n)\}$

Definition:

A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all n and **decreasing** if $a_n > a_{n+1}$ for all n. If it is either increasing or decreasing, it is called **monotonic**.

Definition:

A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leq M$ for all $n \geq 1$ and **bounded below** if there is a number m such that $m \leq a_n$ for all $n \geq 1$. If $\{a_n\}$ is bounded above and below, it is called a **bounded sequence**.

Theorem:

Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent.

Example 6. Show that $\left\{\frac{n}{n^2+1}\right\}$ is convergent.