COMPARISON TESTS

MTH 253 LECTURE NOTES

Exploration: Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$. This is not geometric, but it is *sort of like* the geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$, which is geometric with $a = \frac{1}{2}$ and $r = \frac{1}{2}$ and thus is convergent. We *guess* our series, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$, is also convergent. But how do we justify our guess?

Theorem:

The Comparison Tests
Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent, and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent, and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Example 1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges or diverges. Justify your conclusion as specifically as possible.

Exercise 1. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges or diverges. Justify your conclusion as specifically as possible.

Exploration: Consider $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$. This is not geometric, but it is *sort of like* the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. However, $\frac{1}{2^n-1} \not< \frac{1}{2^n}$, so we cannot use the comparison test.

Theorem:

Limit Comparison Test:
Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $c > 0$ is finite, then either both series converge or both series diverge.

Example 2. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges or diverges. Justify your conclusion as specifically as possible.

Example 3. Determine whether the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt[5]{(n^2+1)^3}}$ converges or diverges. Justify your conclusion as specifically as possible.

Exercise 2. Determine whether the series $\sum_{n=1}^{\infty} sin\frac{1}{n}$ converges or diverges. Justify your conclusion as specifically as possible.