ABSOLUTE CONVERGENCE & THE RATIO TEST

MTH 253 LECTURE NOTES

Definition

A series $\sum a_n$ is **Absolutely Convergent** if $\sum |a_n|$ is convergent.

Definition

A series $\sum a_n$ is **Conditionally Convergent** if $\sum a_n$ is convergent but $\sum |a_n|$ is divergent.

Example 1. Determine whether the alternating harmonic series, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, is absolutely convergent, conditionally convergent, or divergent.

Exercise 1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^3+1}$ is absolutely convergent, conditionally convergent, or divergent.

Theorem

If $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.

Proof:

Note: A series being convergent does *not* imply that it is absolutely convergent (e.g. the alternating harmonic series).

Example 2. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$ is convergent or divergent. Justify your conclusion as specifically as possible.

Technology Exploration: Consider determining whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges or diverges. Explore why the tests learned so far are insufficient. Use Desmos to investigate whether this series appears to converge or diverge.

Theorem

The Ratio Test

Let $\sum a_n$ denote a series and suppose $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

i. If $L < 1$, then $\sum a_n$ is absolutely convergent (and therefore convergent).

ii. If $L > 1$ (or if L is ∞), then $\sum a_n$ diverges.

iii. If $L = 1$, then the Ratio Test is inconclusive.

Note: In the case that you try the Ratio Test and obtain $L = 1$, we then have to try another method! Try a test that we've previously learned.

Example 3. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n}$ is absolutely convergent. Justify your conclusion as specifically as possible.

Example 4. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ converges or diverges. Justify your conclusion as specifically as possible.

Exercise 2. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$ converges or diverges. Justify your conclusion as specifically as possible.

Example 5. Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$ converges or diverges. Justify your conclusion as specifically as possible.

Note: The Ratio Test is best used with series that have a geometric component (like r^n) and/or a factorial component (like $n!$); the test does not work well for series that have only p-series components (like n^p).