POWER SERIES REPRESENTATION

MTH 253 LECTURE NOTES

Exploration: Previously, we saw that

$$g(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$$

has a domain of $(-1, 1)$, due to its properties of being a geometric series. Now, if $x \in (-1, 1)$, what does $g(x)$ converge to (from its properties of being a geometric series)?

We say that the function above is "expressed as the sum of a power series" or "is represented by a power series". Many functions may be expressed as the sum of a power series, and when we express a function in such a way, we must include its interval of convergence.

Example 1. Express $\dfrac{1}{1+x^2}$ as the sum of a power series, and find its interval of convergence.

Technology Exploration: In Desmos, graph $\dfrac{1}{1+x^2}$ along with increasing degrees of partial sums of its power series representation. Does this appear to confirm the interval of convergence?

Example 2. Find a power series representation for $\dfrac{1}{2-x}$, and find its interval of convergence.

Exercise 1. Find a power series representation for $\dfrac{x}{1-x^3}$, and find its interval of convergence.

Theorem

Term-by-Term Differentiation & Integration
If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on $(a-R, a+R)$; moreover,

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\int f(x)\ dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence for each of the derivative and antiderivative of f are both R.

Note: This theorem can be rewritten as such:

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(c_n(x-a)^n\right)$$

$$\int \left(\sum_{n=0}^{\infty} c_n(x-a)^n\right)\ dx = \sum_{n=0}^{\infty} \int \left(c_n(x-a)^n\right)\ dx$$

That is, this theorem states that the sum and difference rules for differentiation and integration over a *finite* sum also hold for power series (which are *infinite* sums); however, we cannot assume that they hold for *all* infinite sums.

Also, note that this theorem states that the derivative and antiderivative of a power series have the same *radius* of convergence; however, they do not necessarily have the same *interval* of convergence – the endpoints must be checked!

Example 3. Let $C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. Use term-by-term differentiation and integration to find power series representations for $C'(x)$ and $\int C(x)\ dx$. What are the intervals of convergence for each of these?

Technology Exploration: We recently discovered graphically that $C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos(x)$. Using Desmos, do the graphs of $C'(x)$ and $\int C(x)\ dx$ follow?

Example 4. Find a power series representation for $\arctan x$ by integrating $\frac{1}{1+x^2}$. What is its interval of convergence?

Exercise 2. Find a power series representation for $\ln(1+x)$, and find its interval of convergence.

Example 5. Evaluate the indefinite integral $\displaystyle\int \frac{1}{1+x^4}\,dx$ as a power series, and find its interval of convergence. Then use this to approximate the definite integral $\displaystyle\int_0^{0.5} \frac{1}{1+x^4}dx$ accurate to six decimal places.