LESSON 4

Velocity and Acceleration

Contents

4.1 Velocity and Acceleration

Past experience with displacement, velocity, and acceleration functions and applications easily extends to motion in space.

Definition 4.1.1

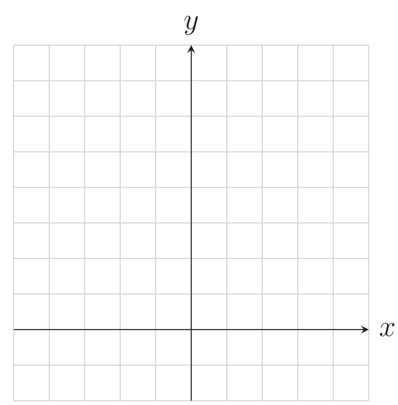
Given the position vector $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$,

the **velocity vector** v is defined as and **acceleration a** is

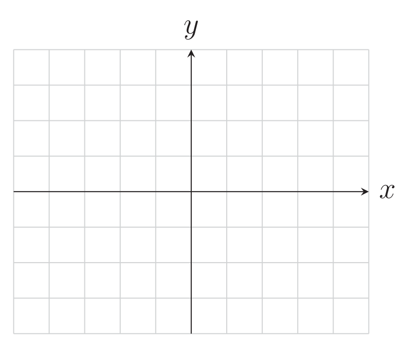
$$\begin{aligned} \mathbf{v}(t) &= \mathbf{r}'(t) \\ &= \left\langle \frac{\mathrm{d}f}{\mathrm{d}t}, \frac{\mathrm{d}g}{\mathrm{d}t}, \frac{\mathrm{d}h}{\mathrm{d}t} \right\rangle \end{aligned}$$

$$\begin{aligned} \mathbf{a}(t) &= \mathbf{v}'(t) \\ &= \left\langle \frac{\mathrm{d}^2 f}{\mathrm{d}t^2}, \frac{\mathrm{d}^2 g}{\mathrm{d}t^2}, \frac{\mathrm{d}^2 h}{\mathrm{d}t^2} \right\rangle. \end{aligned}$$

Example 4.1.1 Find the velocity, acceleration, and speed of a particle with the position function $\mathbf{r}(t) = \langle 2 - t, 4\sqrt{t} \rangle$. Sketch the path of the particle and draw the velocity and acceleration vectors for the specific value of $t = 1$.



Exercise 4.1.1 Find the velocity, acceleration, and speed of a particle with the position function $\mathbf{r}(t) = 3\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j}$. Sketch the path of the particle and draw the velocity and acceleration vectors for the specific value of $t = \frac{\pi}{3}$.



Example 4.1.2 Find the velocity and position vectors of a particle that has an acceleration vector $\mathbf{a}(t) = 2\mathbf{i} + 6t\mathbf{j} + 12t^2\mathbf{k}$ with initial conditions $\mathbf{v}(0) = \mathbf{i}$ and $\mathbf{r}(0) = \mathbf{j} - \mathbf{k}$

Exercise 4.1.2 Find the velocity and position vectors of a particle that has an acceleration vector $\mathbf{a}(t) = \langle -3, 0, -5t \rangle$ with initial conditions $\mathbf{v}(0) = \langle 1, 2, 0 \rangle$ and $\mathbf{r}(0) = \langle 0, 0, 10 \rangle$

4.2 Situational Initial Conditions

Example 4.2.1 A batter hits a baseball 3ft above the ground towards the center field fence which is 10ft high and 400ft from home plate. The ball leaves the bat with speed 115ft/s at an angle of $50°$ above the horizontal. Is it a home run?

Exercise 4.2.1 A ball is thrown at an angle of $45°$ to the ground. If the ball lands 90 meters away, what is its initial speed?

Example 4.2.2 What force is required so that a particle of mass m has the position function $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} - 3t^2\mathbf{k}$

Exercise 4.2.2 A ball with mass 0.8kg is thrown southward into the air with a speed of 30m/s at an angle of 30° to the ground. A west wind applies a steady force of 4N to the ball in an easterly direction. Where does the ball land and with what speed? [Note that acceleration due to gravity is -9.8m/s^2]

4.3 Tangential and Normal Components of Acceleration

In the last lesson we looked at the Unit Tangent Vector Function, the Unit Normal Vector Function, and the curvature of a Vector Valued Function:

Given vector-valued function $\mathbf{r}(t)$, then

- $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$
- $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$
- $\kappa = \dfrac{|\mathbf{T}'|}{|\mathbf{r}'|} = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$

Remember that the Tangent Vector tells us the direction of motion and the Normal Vector tells us the direction of turn. Thus, if we are talking in context of an object moving through space, its acceleration must lie in the plane determined by the Tangent Vector and Normal Vector. This means that we can write the acceleration with components in the tangent and normal directions, which can be useful for some physics applications.

If we rewrite the formula for the Unit Tangent Function as

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{v}}{\nu}$$

implying that $\mathbf{v} = v\mathbf{T}$.

Differentiating both sides we then get

$$\mathbf{a} = \nu'\mathbf{T} + \nu\mathbf{T}'.$$

Now we note that

$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{T}'|}{v}$$

implying that $|\mathbf{T}'| = \kappa\nu$. Thus we get

$$\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|} = \frac{\mathbf{T}'}{\kappa\nu}$$

so that $\mathbf{T}' = \kappa\nu\mathbf{N}$ and thus

$$\mathbf{a} = \nu'\mathbf{T} + \kappa\nu^2\mathbf{N}.$$

If we write

$$\mathbf{a}(t) = a_T\mathbf{T} + a_N\mathbf{N}$$

then we note that

$$a_T = \nu' \text{ and } a_N = \kappa\nu^2.$$

Lemma 4.3.1

$$\nu' = \frac{\mathbf{v}\cdot\mathbf{a}}{\nu} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

Proof : Note that

$$\begin{aligned}\mathbf{v}\cdot\mathbf{a} &= \nu\mathbf{T}\cdot(\nu'\mathbf{T}+\kappa\nu^2\mathbf{N})\\ &= \nu\nu'\mathbf{T}\cdot\mathbf{T}+\kappa\nu^3\mathbf{T}\cdot\mathbf{N}\\ &= \nu\nu'\end{aligned}$$

and thus

$$\nu' = \frac{\mathbf{v}\cdot\mathbf{a}}{\nu} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

with the final equality coming from the definitions of velocity and acceleration.

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Continuing from here we then see that

$$a_T = \nu' = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

and that

$$a_N = \kappa\nu^2 = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}|\mathbf{r}'(t)|^2 = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}.$$

Example 4.3.1 A particle moves with position function $\mathbf{r}(t) = \langle -t^2, t, 2t^3 \rangle$. Determine the tangential and normal components of the acceleration vector.

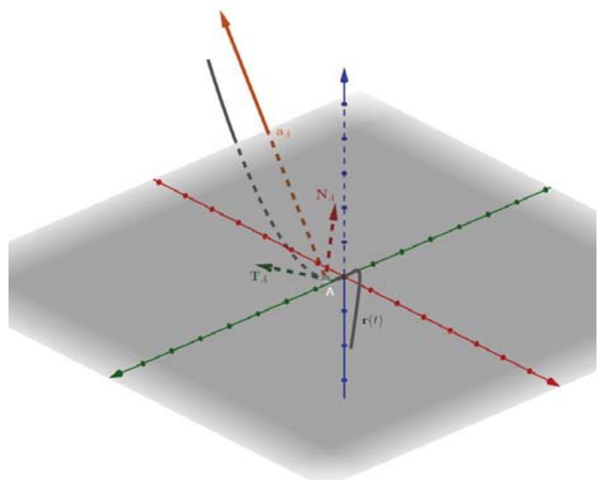

Figure 4.3.1: View Graph Using Geogebra
https://www.geogebra.org/3d/nbp3ar4e

4.4 Minimizing Speed

Exercise 4.4.1 Given a position function of $\mathbf{r}(t) = \langle t^2, \sin(3t), \cos(3t) \rangle$, when is the speed at a minimum?