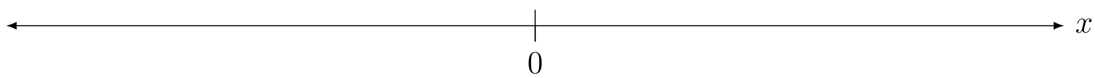
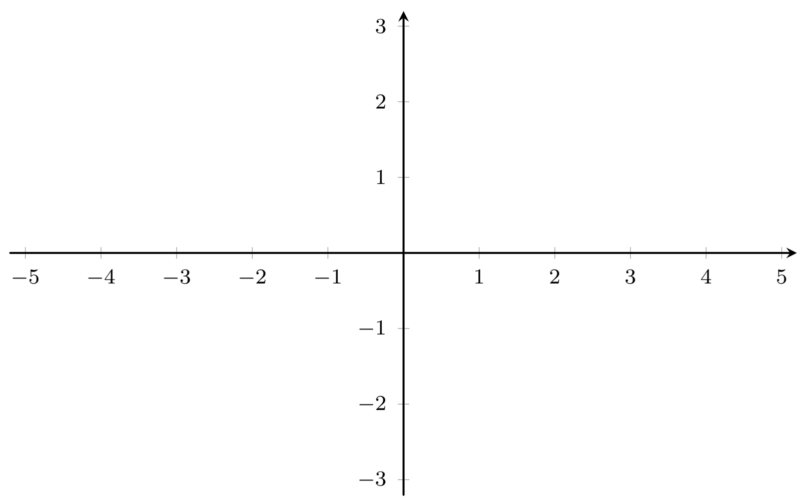
3D COORDINATE SYSTEMS

MTH 253 LECTURE NOTES

Exploration: To graph a single number, we need one line. The **Real Number Line** is drawn by beginning with a point O called the **Origin**, and a single **axis** (line) drawn on it.



To graph the Cartesian plane, we need two real number lines that intersect perpendicularly at the origin.



To graph in three dimensions, we need three real number lines that intersect perpendicularly at the origin.

Definition

To create a **3-dimensional Rectangular Coordinate System**, also known as **space**, **3-space**, or $\mathbb{R}^3$, we begin with a point O called the **Origin**. From O, we draw three directed perpendicular lines called the **coordinate axes**, labelling them the x-, y-, and z-axes using the **Right-Hand Rule** (definition follows).
The xy-plane, xz-plane, and yz-plane are called the **Coordinate Planes** determined by the axes, and these coordinate planes divide $\mathbb{R}^3$ into eight **Octants**.
If P is any point in $\mathbb{R}^3$, then we can represent P with an ordered triple (a, b, c) known as the **Coordinates** of P.

Definition

The **Right-Hand Rule** is a convention used to determine which axis is which, using your right hand. Form your right hand so that your thumb is sticking straight up (as if you are giving someone a "thumbs-up"), your pointer finger is pointing straight ahead (as if you are pointing at something), bend your middle finger 90° (as if your pointer and middle finger are "walking"), and curl your ring and pinky fingers towards you. In this orientation, your extended fingers and thumb represent axes.

Finger	**Axis**
Pointer	x-axis
Middle	y-axis
Thumb	z-axis

Technology Exploration: Use GeoGebra's 3D Graphing tool to explore the axes and the right-hand rule.

Example 1. By hand, draw a set of rectangular coordinate axes for $\mathbb{R}^3$, and label the positive axes. Then plot $P(1, 2, 3)$, $Q(2, -1, 5)$, and $R(4, 1, -2)$. Then, plot P, Q, and R in GeoGebra.

Exploration/Technology Exploration: Consider the following questions. Picture them in your mind, graph them by hand, and graph them in GeoGebra.

- What does $x = 2$ represent in $\mathbb{R}$? In $\mathbb{R}^2$? In $\mathbb{R}^3$?
- What does $x^2 + y^2 = 1$ represent in $\mathbb{R}^2$? In $\mathbb{R}^3$?
- What does $y = x^2$ represent in $\mathbb{R}^2$? In $\mathbb{R}^3$?
- What does $z = 1$ represent?

Exploration: Recall that in $\mathbb{R}^2$, the distance between two points (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This formula is known as the distance formula. This formula is established by drawing a right triangle on the two points and using the Pythagorean Theorem.

Consider $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ as points in $\mathbb{R}^3$. Let's find the distance between P_1 and P_2 in a manner similar to that in $\mathbb{R}^2$.

Theorem

The Distance Formula
The distance $|P_1P_2|$ between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in $\mathbb{R}^3$ is given by

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 2. Find the distance between $P(1, 2, 3)$ and $Q(2, -1, 5)$. Check your answer in GeoGebra.

Exercise 1. Find the distance between $Q(2, -1, 5)$ and $R(4, 1, -2)$. Check your answer in GeoGebra.

Exploration: Let $P(x, y, z)$ be a point on a sphere of a sphere of radius r and center $C(h, k, \ell)$. Find an equation for the sphere.

Theorem

Every point $P(x, y, z)$ on a sphere of radius r and center $C(h, k, \ell)$ satisfies the equation

$$(x-h)^2 + (y-k)^2 + (z-\ell)^2 = r^2$$

Example 3. Find an equation of the unit sphere centered at the origin.

Example 4. Find an equation of a sphere whose center is $P(1, 2, 3)$ and that contains the point $Q(2, -1, 5)$. Graph the sphere in GeoGebra.

Exercise 2. Find an equation of a sphere whose center is $Q(2, -1, 5)$ and that contains the point $R(4, 1, -2)$. Graph the sphere in GeoGebra.

Example 5. Find the center and radius of the sphere $x^2 + y^2 - 4y + z^2 + 2z = 4$. Check your answer in GeoGebra.