LINES

MTH 253 LECTURE NOTES

Exploration: We will explore three different ways to describe a line in 3-space.

Suppose $Q(x_0, y_0, z_0)$ is a point in $\mathbb{R}^3$, $\mathbf{v} \in V_3$ is nonzero, and L is a line through P parallel to $\mathbf{v}$, where $P(x, y, z)$ is an arbitrary point on L.

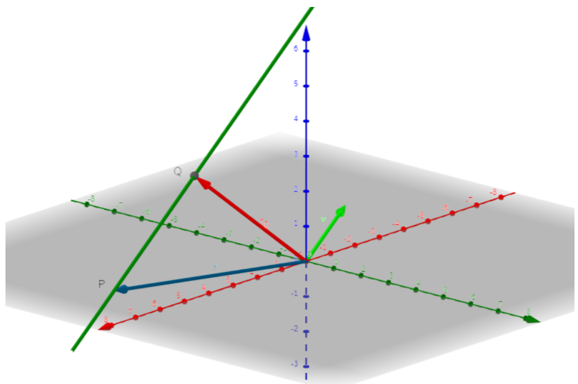

FIGURE 1. https://www.geogebra.org/classic/fsw99hgy

If $\mathbf{r} = \overrightarrow{OP}$ (the position vector of P), $\mathbf{r_0} = \overrightarrow{OQ}$ (position vector of Q), and $\mathbf{a} = \overrightarrow{QP}$, then the Triangle Law of Addition states

$$\mathbf{r} = \mathbf{r_0} + \mathbf{a}$$

Since $\mathbf{a}$ and $\mathbf{v}$ are parallel, $\mathbf{a} = t\mathbf{v}$ for some $t \in \mathbb{R}$. It follows that

Definition

The **Vector Equation** of a line L in $\mathbb{R}^3$ given by

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$$

where $\mathbf{r_0}$ is the position vector of a specific point on L, $\mathbf{v}$ is any vector parallel to L, and t is a **Parameter** which gives $\mathbf{r}$, the position vector of an arbitrary point on L.

Note: When $t > 0$, we get points on one side of Q on the line. When $t < 0$, we get points on the other side.

Exploration: If we write $\mathbf{r} = \langle x, y, z\rangle$, $\mathbf{r_0} = \langle x_0, y_0, z_0\rangle$, and $\mathbf{v} = \langle a, b, c\rangle$, then

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v} \qquad \Longleftrightarrow \qquad \langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$$

Definition

The equations

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

are the **Parametric Equations** of the line L through the point $Q(x_0, y_0, z_0)$ parallel to $\mathbf{v} = \langle a, b, c\rangle$, where $t \in \mathbb{R}$. Each value of t produces a point on L. The numbers a, b, and c are called the **Direction Numbers** of L.

Example 1. Let ℓ be the line parallel to $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ that passes through $P(1, 3, -4)$.

a. Find a vector equation for ℓ.

b. Find parametric equations for ℓ.

c. Find four points on ℓ.

d. Use GeoGebra to graph ℓ.

Definition

If the direction numbers for a line L are all nonzero, then the **Symmetric Equations** of L are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

If $a = 0$, then the symmetric equations of L are

$$x = x_0 \quad , \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Similar symmetric equations hold when $b = 0$ and/or $c = 0$.

Note: Symmetric equations for a line are found by eliminating the parameter from parametric equations.

Example 2. Let ℓ be the line parallel to $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ that passes through $P(1, 3, -4)$. Find the symmetric equations for ℓ.

Example 3. Let ℓ be the line parallel to $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ that passes through $P(1, 3, -4)$. Find the points at which ℓ intersects each of the xy-, xz-, and yz-planes.

Exercise 1. Let ℓ be the line through the points $P(1, 2, 3)$ and $Q(2, -1, 5)$. Find the following.

a. A vector equation for ℓ.

b. Parametric equations for ℓ.

c. Symmetric equations for ℓ.

d. The point at which ℓ intersects each of the xy-, xz-, and yz-planes.

Definition

Two nonintersecting lines that are not parallel are called **Skew Lines**.

Example 4. Show that the lines ℓ_1 and ℓ_2 whose parametric equations are

$$\begin{aligned} \ell_1 : \quad & x = 1 + 2t \quad & y = 3 - t \quad & z = -4 + 5t \\ \ell_2 : \quad & x = 1 + s \quad & y = 2 - 3s \quad & z = 3 + 2s \end{aligned}$$

are skew lines.

Exercise 2. Let L_1 be the line parallel to $\langle 1, 2, 3 \rangle$ passing through the point $(2, -1, 5)$. Let L_2 be the line whose symmetric equations are

$$\frac{x - 2}{2} = y + 1 = \frac{z + 1}{3}.$$

Determine whether L_1 are and L_2 are parallel, skew, or intersecting lines.